Math 245B Lecture 12 Notes

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1 Banach Space Constructions

1.1 Product spaces

Definition 1.1. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed vector spaces over K. The **Cartesian product** $\mathcal{X} \times \mathcal{Y}$ is a normed space with one of many possible norms:

1. $\|(x, y)\| := \max(|x\|_{\mathcal{X}}, \|y\|_{\mathcal{Y}})$
2. $\|(x, y)\| := |x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$
3. $\|(x, y)\| := \sqrt{\|x\|_{\mathcal{X}}^2 + \|y\|_{\mathcal{Y}}^2}$.

Remark 1.1. There are many natural options for what norm to use; not all of them are listed here. However, from a category theory perspective, none of these are "natural."

Proposition 1.1. *With any of these norms, $\mathcal{X} \times \mathcal{Y}$ is complete if and only if both $\mathcal{X}$ and $\mathcal{Y}$ are complete.*

1.2 Quotient spaces

Definition 1.2. Let $(\mathcal{X}, \|\cdot\|)$ be a normed space over K, and let $\mathcal{M} \subseteq \mathcal{X}$ be a vector subspace. The **quotient space** is $\mathcal{X}/\mathcal{M} = \{x + \mathcal{M} : x \in \mathcal{X}\}$ with the **quotient norm** $\|x + \mathcal{M}\| := \inf\{\|y\| : y \in x + \mathcal{M}\}$.

Lemma 1.1. *If $\mathcal{X}$ is complete and $\mathcal{M} \subseteq \mathcal{X}$ is a closed subspace, then $\mathcal{X}/\mathcal{M}$ is complete.*

Proof. Suppose $(x_n + \mathcal{M})_{n=1}^\infty \in \mathcal{X}/\, cM$ is a sequence such that $\sum_{n=1}^\infty \|x_n + \mathcal{M}\|$. For each n, pick $y_n \in x_n + \mathcal{M}$ such that $\|y_n\| < \|x_n + \mathcal{M}\| + 2^{-n}$. Then $\sum_{n=1}^\infty \|y_n\| < \infty$, so there exists some $y = \sum_{n=1}^\infty y_n \in \mathcal{X}$. So $\|y - \sum_{n=1}^N \| \to 0$ as $N \to \infty$. This is an element of $(y + \mathcal{M}) - \sum_{n=1}^N (y_n + M) = (y + \mathcal{M}) - \sum_{n=1}^N (x_n + M)$. So

$$\left\|(y + \mathcal{M}) - \sum_{n=1}^N (x_n + M)\right\| \leq \left\|y - \sum_{n=1}^N\right\| \to 0. \qquad \square$$

1.3 Bounded linear maps

Definition 1.3. A linear map $T : \mathcal{X} \to \mathcal{Y}$ is called **bounded** if there exists some $C < \infty$ such that $\|T_x\|_{\mathcal{Y}} \leq C\|x\|_{\mathcal{X}}$ for all $x \in \mathcal{X}$. The **vector space of bounded linear maps** is called $\mathcal{L}(\mathcal{X}, \mathcal{Y})$.

Proposition 1.2. *Let $T : \mathcal{X} \to \mathcal{Y}$ be linear. The following are equivalent:*

1. *T is continuous.*
2. *T is continuous at 0.*
3. *T is bounded.*

Proof. (1) $\implies$ (2): This is a special case.

(3) $\implies$ (1): For all $x, x' \in \mathcal{X}$, we have

$$\|Tx - Tx'\|_{\mathcal{Y}} = \|T(x - x')\|_{\mathcal{Y}} \leq C\|x - x'\|_{\mathcal{X}}.$$

(2) $\implies$ (3): For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|x\|_{\mathcal{X}} < \delta \implies \|T_x\|_{\mathcal{Y}} < \varepsilon.$$

So for all $x \in \mathcal{X} \setminus \{0\}$, let $x' = \frac{\delta}{2\|x\|_{\mathcal{X}}}$. Then $\|x'\|_{\mathcal{X}} < \delta$. Then

$$\|Tx'\|_{\mathcal{Y}} = \frac{\delta}{2\|x\|_{\mathcal{X}}}\|Tx\|_{\mathcal{Y}} < \varepsilon \implies \|T_x\|_{\mathcal{Y}} < \left(\frac{2\varepsilon}{\delta}\right)\|x\|_{\mathcal{X}}.$$ □

Lemma 1.2. *If $S, T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, say with constants C_S, C_T, then $S + T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ with constant at most $C_S + C_T$, and $\lambda S\mathcal{L}(\mathcal{X}, \mathcal{Y})$ with constant $\leq |\lambda|C_S$*

Proof.

$$\|(S + T)x\| \leq \|Sx\| + \|Tx\| \leq (C_S + C_T)\|x\|.$$ □

Definition 1.4. $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a normed space with the **operator norm**

$$\|T\|_{\text{op}} = \inf\{C : \|Tx\| \leq C\|x\| \, \forall x \in X\}.$$

Remark 1.2. Equivalently, we can define the operator norm as

$$\begin{aligned}\|T\|_{\text{op}} &= \sup\{C : \|Tx\|_{\mathcal{Y}} : x \in \mathcal{X}, \|x\|_{\mathcal{X}} = 1\}.\\ &= \sup\left\{C : \frac{\|Tx\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}} : x \in \mathcal{X} \setminus \{0\}\right\}.\end{aligned}$$

Proposition 1.3. *If Y is complete, so is $\mathcal{L}(\mathcal{X}, \mathcal{Y})$.*

Proof. Let $(T_n)_n$ be Cauchy in $\mathcal{L}(\mathcal{X},\mathcal{Y})$. Then for all $x \in \mathcal{X}$, we have

$$\|T_n x - T_m x\|_{\mathcal{Y}} \leq \|T_n - T_m\|_{\text{op}} \|x\|_{\mathcal{X}} \xrightarrow{n,m\to\infty} 0,$$

so there exists a $\lim_n T_n x =: Tx$. Now show that $T \in L(\mathcal{X},\mathcal{Y})$,and $\|T_n - T\|_{\text{op}} \to 0$. $\square$

Remark 1.3. If $S \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{Y},\mathcal{Z})$, then for all $x \in \mathcal{X}$,

$$\|TSx\|_{\mathcal{Z}} = \|T\|\|Sx\|_{\mathcal{Y}} \leq \|T\|\|S\|\|x\|_{\mathcal{X}},$$

so $T \circ S \in \mathcal{L}(\mathcal{X},\mathcal{Z})$, and $\|T \circ S\| \leq \|S\|\|T\|$. So $L(\mathcal{X},\mathcal{X})$ is an algebra over $\mathcal{K}$, and it is a Banach algebra if $\mathcal{X}$ is complete.

Definition 1.5. A linear operator $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ is **invertible** (or an **isomorphism**) if T^{-1} exists an and is an element of $\mathcal{L}(\mathcal{Y},\mathcal{X})$.

1.4 Dual spaces and the Hahn-Banach theorem

Definition 1.6. The space $\mathcal{X}^* := \mathcal{L}(\mathcal{X}, K)$ is the **dual space**. Its norm is called the **dual norm**, and its elements are **bounded linear functionals**.

Theorem 1.1 (Hahn-Banach)**.** *Let $(\mathcal{X}, \|\cdot\|)$ be a normed space, let $\mathcal{M}$ be a linear subspace, and let $f \in \mathcal{M}^*$. Then there exists $F \in \mathcal{X}^*$ such that $F|_{\mathcal{M}} = f$ and $\|F\|_{\mathcal{X}^*} = \|f\|_{\mathcal{M}^*}$.*

We will prove this theorem next time. Instead, let's look at a consequence.

Theorem 1.2. *If $\mathcal{M} \subseteq \mathcal{X}$ is a closed linear subspace and $x \in \mathcal{X} \setminus \mathcal{M}$, then there exists $f \in \mathcal{X}^*$ such that $f|_{\mathcal{M}} = 0$ but $f(x) \neq 0$. Moreover, we can take $\|f\| = 1$ and $f(x) = \inf_{y\in\mathcal{M}} \|x - y\|$.*

Proof. Let $\mathcal{N} = \mathcal{M} + Kx$. Let $\delta = \inf_{y\in M} \|x - y\| = \delta$. Define the function $g : \mathcal{N} \to K$ as $g(y + \lambda x) := 0 + \lambda\delta$. To show that g is well-defined and linear, note that

$$g((y + \lambda x) + (y' + \lambda' x)) = g((y + y') + (\lambda + \lambda')x) = (\lambda + \lambda')\delta.$$

For find the norm of g, we want $|g(y + \lambda x)| \leq \|y + x\|$ for all y, λ. Scaling by a constant, we can assume $\lambda = 1$. Then we want $\delta = |g(y + x)| \leq \|y + x\|$ for all $y \in \mathcal{M}$, which is true by definition. By the Hahn-Banach theorem, g has an extension $f \in \mathcal{X}^*$ with $\|f\| = \|g\| = 1$. $\square$